Operator algebras and conjugacy problem for the pseudo-Anosov automorphisms of a surface

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Abstract

The conjugacy problem for the pseudo-Anosov automorphisms of a compact surface is studied. To each pseudo-Anosov automorphism ϕ, we assign an AF-algebra $\mathbb{A}_\phi$ (an operator algebra). It is proved that the assignment is functorial, i.e. every ϕ', conjugate to ϕ, maps to an AF-algebra $\mathbb{A}_{\phi'}$, which is stably isomorphic to $\mathbb{A}_\phi$. The new invariants of the conjugacy of the pseudo-Anosov automorphisms are obtained from the known invariants of the stable isomorphisms of the AF-algebras. Namely, the main invariant is a triple $(\Lambda, [I], K)$, where Λ is an order in the ring of integers in a real algebraic number field K and $[I]$ an equivalence class of the ideals in Λ. The numerical invariants include the determinant Δ and the signature Σ, which we compute for the case of the Anosov automorphisms. A question concerning the p-adic invariants of the pseudo-Anosov automorphism is formulated.

Key words and phrases: mapping class group, AF-algebras

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1 Introduction

A. Conjugacy problem. Let $Mod\ (X)$ be the mapping class group of a compact surface X, i.e. the group of the orientation preserving automorphisms of X modulo the trivial ones. Recall that $\phi, \phi' \in Mod\ (X)$ are the conjugate automorphisms, whenever $\phi' = h\circ\phi\circ h^{-1}$ for an $h \in Mod\ (X)$. It is not hard to see that the conjugation is an equivalence relation which splits the mapping class group into the disjoint classes of conjugate automorphisms. The invariants of the conjugacy classes in $Mod\ (X)$ is an important and difficult problem studied

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by Hemion [8], Mosher [11] and others. Knowing such invariants leads to a topological classification of the three-dimensional manifolds, which fiber over the circle with the monodromy $\phi \in Mod\ (X)$ [14].

B. The pseudo-Anosov automorphisms. It is known that any $\phi \in Mod\ (X)$ is isotopic to an automorphism ϕ', such that either (i) ϕ' has a finite order, or (ii) ϕ' is a pseudo-Anosov (aperiodic) automorphism, or else (iii) ϕ' is reducible by a system of curves Γ surrounded by the small tubular neighborhoods $N(\Gamma)$, such that on $X - N(\Gamma)$ ϕ' satisfies either (i) or (ii). In the case $\phi \in Mod\ (X)$ is a pseudo-Anosov automorphism, there exist a pair of the stable $\mathcal{F}_s$ and unstable $\mathcal{F}_u$ mutually orthogonal measured foliations on the surface X, such that $\phi(\mathcal{F}_s) = \frac{1}{\lambda_\phi}\mathcal{F}_s$ and $\phi(\mathcal{F}_u) = \lambda_\phi \mathcal{F}_u$, where $\lambda_\phi > 1$ is called a dilatation of ϕ. The foliations $\mathcal{F}_s, \mathcal{F}_u$ are minimal, uniquely ergodic and describe the automorphism ϕ up to a power. In the sequel, we shall focus on the conjugacy problem for the pseudo-Anosov automorphisms of a surface X.

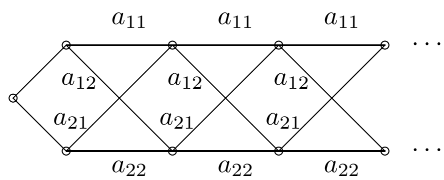
C. The AF-algebras. The C^*-algebra is an algebra A over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \le ||a||\ ||b||$ and $||a^*a|| = ||a^2||$ for all $a, b \in A$. The C^*-algebras have been introduced by Murray and von Neumann as the rings of bounded operators on a Hilbert space and are deeply intertwined with the geometry and topology of manifolds [3], §24. Any simple finite-dimensional C^*-algebra is isomorphic to the algebra $M_n(\mathbb{C})$ of the complex $n \times n$ matrices. A natural completion of the finite-dimensional semi-simple C^*-algebras (as $n \to \infty$) is known as an *AF-algebra* [6]. The AF-algebra is most conveniently given by an infinite graph, which records the inclusion of the finite-dimensional subalgebras into the AF-algebra. The graph is called a *Bratteli diagram.* In an important special case when the diagram is periodic, the AF-algebra is called stationary. In the addition to a regular isomorphism $\cong$, the C^*-algebras A, A' are called *stably isomorphic* whenever $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators.

D. Main idea. Let $\phi \in Mod\ (X)$ be a pseudo-Anosov automorphism. The main idea of the present paper is to assign to ϕ an AF-algebra, $\mathbb{A}_\phi$, so that for every $h \in Mod\ (X)$ the following diagram commutes:

$$\begin{tikzcd}
\phi \arrow[r, "\text{conjugacy}"] \arrow[d] & \phi' = h \circ \phi \circ h^{-1} \arrow[d] \\
\mathbb{A}_\phi \otimes \mathcal{K} \arrow[r, "\text{isomorphism}"] & \mathbb{A}_{\phi'} \otimes \mathcal{K}
\end{tikzcd}$$

(In other words, if ϕ, ϕ' are the conjugate pseudo-Anosov automorphisms, then the AF-algebras $\mathbb{A}_\phi, \mathbb{A}_{\phi'}$ are stably isomorphic.) For the sake of clarity, we shall consider an example illustrating the idea in the case $X = T^2$ (a torus).

E. Model example. Let $\phi \in Mod\ (T^2)$ be the Anosov automorphism given by a non-negative matrix $A_\phi \in SL_2(\mathbb{Z})$. Consider a stationary AF-algebra, $\mathbb{A}_\phi$, given by the following periodic Bratteli diagram:



$$A_\phi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Figure 1: The AF-algebra $\mathbb{A}_\phi$.

where a_{ij} indicate the multiplicity of the respective edges of the graph. (We encourage the reader to verify that $F : \phi \mapsto \mathbb{A}_\phi$ is a correctly defined function on the set of Anosov automorphisms given by the hyperbolic matrices with the non-negative entries.) Let us show that if $\phi, \phi' \in Mod\ (T^2)$ are the conjugate Anosov automorphisms, then $\mathbb{A}_\phi, \mathbb{A}_{\phi'}$ are the stably isomorphic AF-algebras. Indeed, let $\phi' = h \circ \phi \circ h^{-1}$ for an $h \in Mod\ (X)$. Then $A_{\phi'} = TA_\phi T^{-1}$ for a matrix $T \in SL_2(\mathbb{Z})$. Note that $(A'_\phi)^n = (TA_\phi T^{-1})^n = TA_\phi^n T^{-1}$, where $n \in \mathbb{N}$. We shall use the following criterion ([6], Theorem 2.3): the AF-algebras $\mathbb{A}, \mathbb{A}'$ are stably isomorphic if and only if their Bratteli diagrams contain a common block of an arbitrary length. Consider the following sequences of matrices:

$$\left\{ \begin{array}{c} \underbrace{A_\phi A_\phi \dots A_\phi}_{n} \\ T \underbrace{A_\phi A_\phi \dots A_\phi}_{n} T^{-1}, \end{array} \right.$$

which mimic the Bratteli diagrams of $\mathbb{A}_\phi$ and $\mathbb{A}_{\phi'}$. Letting $n \to \infty$, we conclude that $\mathbb{A}_\phi \otimes \mathcal{K} \cong \mathbb{A}_{\phi'} \otimes \mathcal{K}$.

F. The invariants of torus automorphisms obtained from the operator algebras. The conjugacy problem for the Anosov automorphisms can now be recast in terms of the AF-algebras: find the invariants of the stable isomorphism classes of the stationary AF-algebras. One such invariant is due to Handelman [7]. Consider an eigenvalue problem for the hyperbolic matrix $A_\phi \in SL_2(\mathbb{Z})$: $A_\phi v_A = \lambda_A v_A$, where $\lambda_A > 1$ is the Perron-Frobenius eigenvalue and $v_A = (v_A^{(1)}, v_A^{(2)})$ the corresponding eigenvector with the positive entries normalized

so that $v_A^{(i)} \in K = \mathbb{Q}(\lambda_A)$. Denote by $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \mathbb{Z}v_A^{(2)}$ a $\mathbb{Z}$-module in the number field K. Recall that the coefficient ring, Λ, of module $\mathfrak{m}$ consists of the elements $\alpha \in K$ such that $\alpha\mathfrak{m} \subseteq \mathfrak{m}$. It is known that Λ is an order in K (i.e. a subring of K containing 1) and, with no restriction, one can assume that $\mathfrak{m} \subseteq \Lambda$. It follows from the definition, that $\mathfrak{m}$ coincides with an ideal, I, whose equivalence class in Λ we shall denote by $[I]$. It has been proved by Handelman, that the triple $(\Lambda, [I], K)$ is an arithmetic invariant of the stable isomorphism class of $\mathbb{A}_\phi$: the $\mathbb{A}_\phi, \mathbb{A}_{\phi'}$ are stably isomorphic AF-algebras if and only if $\Lambda = \Lambda', [I] = [I']$ and $K = K'$. It is interesting to compare the operator algebra invariants with those obtained in [16].

G. The AF-algebra $\mathbb{A}_\phi$ (pseudo-Anosov case). Denote by $\mathcal{F}_\phi$ the stable foliation of a pseudo-Anosov automorphism $\phi \in Mod\ (X)$. For brevity, we assume that $\mathcal{F}_\phi$ is an oriented foliation given by the trajectories of a closed 1-form $\omega \in H^1(X; \mathbb{R})$. Let $v^{(i)} = \int_{\gamma_i} \omega$, where $\{\gamma_1, \ldots, \gamma_n\}$ is a basis in the relative homology $H_1(X, Sing\ \mathcal{F}_\phi; \mathbb{Z})$ and denote by $\theta = (\theta_1, \ldots, \theta_{n-1})$ a vector with the coordinates $\theta_i = v^{(i+1)}/v^{(1)}$. Consider the (infinite) Jacobi-Perron continued fraction [2] of θ:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where $b_i = (b_1^{(i)}, \ldots, b_{n-1}^{(i)})^T$ is a vector of the non-negative integers, I the unit matrix and $\mathbb{I} = (0, \ldots, 0, 1)^T$. By the definition, $\mathbb{A}_\phi$ is an AF-algebra given by the Bratteli diagram, whose incidence matrices coincide with $B_k = \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}$ for $k = 1, \ldots, \infty$.

H. Main results. For a matrix $A \in GL_n(\mathbb{Z})$ with the positive entries, we let λ_A be the Perron-Frobenius eigenvalue and $(v_A^{(1)}, \ldots, v_A^{(n)})$ the corresponding normalized eigenvector with $v_A^{(i)} \in K = \mathbb{Q}(\lambda_A)$. The coefficient (endomorphism) ring of the module $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \ldots + \mathbb{Z}v_A^{(n)}$ will be denoted by Λ. The equivalence class of ideals in the ring Λ generated by the ideal $\mathfrak{m}$, we shall write as $[I]$. Finally, we denote by $\Delta = det\ (a_{ij})$ and Σ the determinant and signature of the symmetric bilinear form $q(x, y) = \sum_{i,j}^n a_{ij} x_i x_j$, where $a_{ij} = Tr\ (v_A^{(i)} v_A^{(j)})$ and $Tr(\bullet)$ the trace function. Our main results can be expressed as follows.

Theorem 1 *$\mathbb{A}_\phi$ is a stationary AF-algebra.*

Theorem 2 *Let F be a mapping acting by the formula $\phi \mapsto \mathbb{A}_\phi$. Then:*

(i) F is a functor, i.e. F maps the conjugate pseudo-Anosov automorphisms to the stably isomorphic AF-algebras;

(ii) $Ker\ F = [\phi]$, where $[\phi] = \{\phi' \in Mod\ (X) \mid (\phi')^m = \phi^n,\ m, n \in \mathbb{N}\}$ is the commensurability class of the pseudo-Anoov automorphism ϕ.

Corollary 1 *The following are the invariants of the conjugacy classes of the pseudo-Anosov automorphisms:*

(i) triples $(\Lambda, [I], K)$*;*

(ii) integers Δ *and* Σ*.*

I. Structure of the paper. The proof of main results can be found in section 4. The sections 2 and 3 are reserved for some helpful lemmas necessary to prove the main results. Finally, in section 5 some examples, open problems and conjectures are considered. Since the paper does not include a formal section on the preliminaries, we encourage the reader to consult [3], [6], [10] (operator algebras & dynamics), [9], [15] (measured foliations) and [2], [12] (Jacobi-Perron continued fractions).

2 The Jacobian of a measured foliation

Let $\mathcal{F}$ be a measured foliation on a compact surface X [15]. For the sake of brevity, we shall always assume that $\mathcal{F}$ is an oriented foliation, i.e. given by the trajectories of a closed 1-form ω on X. (The assumption is no restriction – each measured foliation is oriented on a surface $\widetilde{X}$, which is a double cover of X ramified at the singular points of the half-integer index of the non-oriented foliation [9].) Let $\{\gamma_1, \dots, \gamma_n\}$ be a basis in the relative homology group $H_1(X, Sing\ \mathcal{F}; \mathbb{Z})$, where $Sing\ \mathcal{F}$ is the set of singular points of the foliation $\mathcal{F}$. It is well known that $n = 2g + m - 1$, where g is the genus of X and $m = |Sing\ (\mathcal{F})|$. The periods of ω in the above basis we shall write as:

$$\lambda_i = \int_{\gamma_i} \omega.$$

The λ_i are the coordinates of $\mathcal{F}$ in the space of all measured foliations on X (with a fixed set of the singular points) [5].

Definition 1 *By a Jacobian Jac* $(\mathcal{F})$ *of the measured foliation* $\mathcal{F}$*, we understand a* $\mathbb{Z}$*-module* $\mathfrak{m} = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_n$ *regarded as a subset of the real line* $\mathbb{R}$*.*

An importance of the Jacobians stems from an observation that although the periods λ_i depend on the basis in $H_1(X, Sing\ \mathcal{F}; \mathbb{Z})$, the Jacobian does not. Moreover, up to a scalar multiple, the Jacobian is an invariant of the equivalence class of the foliation $\mathcal{F}$. We shall formalize the observations in the following two lemmas.

Lemma 1 *The* $\mathbb{Z}$*-module* $\mathfrak{m}$ *is independent of the choice of a basis in* $H_1(X, Sing\ \mathcal{F}; \mathbb{Z})$ *and depends solely on the foliation* $\mathcal{F}$*.*

Proof. Indeed, let $A = (a_{ij}) \in GL_n(\mathbb{Z})$ and let

$$\gamma'_i = \sum_{j=1}^{n} a_{ij} \gamma_j$$

be a new basis in $H_1(X, Sing\ \mathcal{F}; \mathbb{Z})$. Then using the integration rules:

$$\begin{array}{lll} \lambda'_i & = \int_{\gamma'_i} \omega & = \displaystyle\int_{\sum_{j=1}^{n} a_{ij}\gamma_j} \omega = \\ & = \sum_{j=1}^{n} \int_{\gamma_j} \omega & = \displaystyle\sum_{j=1}^{n} a_{ij} \lambda_j. \end{array}$$

To prove that $\mathfrak{m} = \mathfrak{m}'$, consider the following equations:

$$\begin{array}{lll} \mathfrak{m}' & = \sum_{i=1}^{n} \mathbb{Z} \lambda'_i & = \displaystyle\sum_{i=1}^{n} \mathbb{Z} \sum_{j=1}^{n} a_{ij} \lambda_j = \\ & = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} \mathbb{Z} \right) \lambda_j & \subseteq \mathfrak{m}. \end{array}$$

Let $A^{-1} = (b_{ij}) \in GL_n(\mathbb{Z})$ be an inverse to the matrix A. Then $\lambda_i = \sum_{j=1}^{n} b_{ij} \lambda'_j$ and

$$\begin{array}{lll} \mathfrak{m} & = \sum_{i=1}^{n} \mathbb{Z} \lambda_i & = \displaystyle\sum_{i=1}^{n} \mathbb{Z} \sum_{j=1}^{n} b_{ij} \lambda'_j = \\ & = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_{ij} \mathbb{Z} \right) \lambda'_j & \subseteq \mathfrak{m}'. \end{array}$$

Since both $\mathfrak{m}' \subseteq \mathfrak{m}$ and $\mathfrak{m} \subseteq \mathfrak{m}'$, we conclude that $\mathfrak{m}' = \mathfrak{m}$. Lemma 1 follows. $\square$

Recall that the measured foliations $\mathcal{F}$ and $\mathcal{F}'$ are said to be *equivalent*, if there exists an automorphism $h \in Mod\ (X)$, which sends the leaves of the foliation $\mathcal{F}$ to the leaves of the foliation $\mathcal{F}'$. Note that the equivalence deals with the topological foliations (i.e. the projective classes of measured foliations [15]) and does not preserve a transversal measure of the leaves.

Lemma 2 *Let $\mathcal{F}, \mathcal{F}'$ be the equivalent measured foliations on a surface X. Then*

$$Jac\ (\mathcal{F}') = \mu\ Jac\ (\mathcal{F}),$$

where $\mu > 0$ is a real number.

Proof. Let $h : X \to X$ be an automorphism of the surface X. Denote by h_* its action on $H_1(X, Sing\ (\mathcal{F}); \mathbb{Z})$ and by h^* on $H^1(X; \mathbb{R})$ connected by the formula:

$$\int_{h_*(\gamma)} \omega = \int_{\gamma} h^*(\omega), \qquad \forall \gamma \in H_1(X, Sing\ (\mathcal{F}); \mathbb{Z}), \qquad \forall \omega \in H^1(X; \mathbb{R}).$$

Let $\omega, \omega' \in H^1(X; \mathbb{R})$ be the closed 1-forms whose trajectories define the foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. Since $\mathcal{F}, \mathcal{F}'$ are equivalent measured foliations,

$$\omega' = \mu \ h^*(\omega)$$

for a $\mu > 0$.

Let $Jac\ (\mathcal{F}) = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $Jac\ (\mathcal{F}') = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$. Then:

$$\lambda'_i = \int_{\gamma_i} \omega' = \mu \int_{\gamma_i} h^*(\omega) = \mu \int_{h_*(\gamma_i)} \omega, \qquad 1 \leq i \leq n.$$

By lemma 1, it holds:

$$Jac\ (\mathcal{F}) = \sum_{i=1}^{n} \mathbb{Z} \int_{\gamma_i} \omega = \sum_{i=1}^{n} \mathbb{Z} \int_{h_*(\gamma_i)} \omega.$$

Therefore:

$$Jac\ (\mathcal{F}') = \sum_{i=1}^{n} \mathbb{Z} \int_{\gamma_i} \omega' = \mu \sum_{i=1}^{n} \mathbb{Z} \int_{h_*(\gamma_i)} \omega = \mu\ Jac\ (\mathcal{F}).$$

Lemma 2 follows. $\square$

3 Equivalent foliations are stably isomorphic

Let $\mathcal{F}$ be a measured foliation on the surface X. We introduce an AF-algebra, $\mathbb{A}_{\mathcal{F}}$, of the foliation $\mathcal{F}$ the same way it was done in §1.G for the foliation $\mathcal{F}_\phi$. The goal of present section is the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\text{equivalent}"] \arrow[d] & \mathcal{F}' \arrow[d] \\
\mathbb{A}_{\mathcal{F}} \otimes \mathcal{K} \arrow[r, "\text{isomorphic}"] & \mathbb{A}_{\mathcal{F}'} \otimes \mathcal{K}
\end{tikzcd}$$

Let us start with a lemma, which is a simple property of the Jacobi-Perron fractions [2].

Lemma 3 *Let $\mathfrak{m} = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $\mathfrak{m}' = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$ be two $\mathbb{Z}$-modules, such that $\mathfrak{m}' = \mu\mathfrak{m}$ for a $\mu > 0$. Then the Jacobi-Perron continued fractions of the vectors λ and λ' coincide except, may be, a finite number of terms.*

Proof. Let $\mathfrak{m} = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $\mathfrak{m}' = \mathbb{Z}\lambda_1' + \ldots + \mathbb{Z}\lambda_n'$. Since $\mathfrak{m}' = \mu\mathfrak{m}$, where μ is a positive real, one gets the following identity of the $\mathbb{Z}$-modules:

$$\mathbb{Z}\lambda_1' + \ldots + \mathbb{Z}\lambda_n' = \mathbb{Z}(\mu\lambda_1) + \ldots + \mathbb{Z}(\mu\lambda_n).$$

One can always assume that λ_i and λ_i' are positive reals. For obvious reasons, there exists a basis $\{\lambda_1^{''}, \ldots, \lambda_n^{''}\}$ of the module $\mathfrak{m}'$, such that:

$$\begin{cases} \lambda'' & = A(\mu\lambda) \\ \lambda'' & = A'\lambda', \end{cases}$$

where $A, A' \in GL_n^+(\mathbb{Z})$ are the matrices, whose entries are non-negative integers. In view of the Proposition 3 of [1]:

$$\begin{cases} A & = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \\ A' & = \begin{pmatrix} 0 & 1 \\ I & b_1' \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_l' \end{pmatrix}, \end{cases}$$

where b_i, b_i' are non-negative integer vectors. Since that (Jacobi-Perron) continued fraction for the vectors λ and $\mu\lambda$ coincide for any $\mu > 0$ [2], we conclude that:

$$\begin{cases} \begin{pmatrix} 1 \\ \theta \end{pmatrix} & = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} \\ \begin{pmatrix} 1 \\ \theta' \end{pmatrix} & = \begin{pmatrix} 0 & 1 \\ I & b_1' \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_l' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \end{cases}$$

where

$$\begin{pmatrix} 1 \\ \theta'' \end{pmatrix} = \lim_{i\to\infty} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & a_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}.$$

In other words, the continued fractions of the vectors λ and λ' coincide, except a finite number of terms. $\square$

Lemma 4 *Let $\mathcal{F}, \mathcal{F}'$ be the equivalent measured foliations on a surface X. Then the AF-algebras $\mathbb{A}_{\mathcal{F}}, \mathbb{A}_{\mathcal{F}'}$ are stably isomorphic.*

Proof. Notice that lemma 2 implies that the equivalent measured foliations $\mathcal{F}, \mathcal{F}'$ have the proportional Jacobians, i.e. $\mathfrak{m}' = \mu\mathfrak{m}$ for a $\mu > 0$. On the other hand, by lemma 3 the continued fraction expansion of the basis vectors of the proportional Jacobians must coincide, except a finite number of terms. Thus, the AF-algebras $\mathbb{A}_{\mathcal{F}}$ and $\mathbb{A}_{\mathcal{F}'}$ are given by the Bratteli diagrams, which are identical, except a finite part of the diagram. It is well known ([6], Theorem 2.3) that the AF-algebras, which have such a property, are stably isomorphic. $\square$

4 Proofs

4.1 Proof of Theorem 1

Let $\phi \in Mod\ (X)$ be a pseudo-Anosov automorphism of the surface X. Denote by $\mathcal{F}_\phi$ the invariant foliation of ϕ. By the definition of such a foliation, $\phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi$, where $\lambda_\phi > 1$ is the dilatation of ϕ.

Consider the Jacobian $Jac\ (\mathcal{F}_\phi) = \mathfrak{m}_\phi$ of the measured foliation $\mathcal{F}_\phi$. Since $\mathcal{F}_\phi$ is an invariant foliation of the pseudo-Anosov automorphism ϕ, one gets the following equality of the $\mathbb{Z}$-modules:

$$\mathfrak{m}_\phi = \lambda_\phi \mathfrak{m}_\phi, \qquad \lambda_\phi \neq \pm 1. \tag{1}$$

Let $\{v^{(1)}, \dots, v^{(n)}\}$ be a basis in the module $\mathfrak{m}_\phi$, such that $v^{(i)} > 0$. In view of (1), one obtains the following system of linear equations:

$$\left\{\begin{array}{lll} \lambda_\phi v^{(1)} & = & a_{11} v^{(1)} + a_{12} v^{(2)} + \dots + a_{1n} v^{(n)} \\ \lambda_\phi v^{(2)} & = & a_{21} v^{(1)} + a_{22} v^{(2)} + \dots + a_{2n} v^{(n)} \\ \quad\vdots & & \\ \lambda_\phi v^{(n)} & = & a_{n1} v^{(1)} + a_{n2} v^{(2)} + \dots + a_{nn} v^{(n)}, \end{array}\right. \tag{2}$$

where $a_{ij} \in \mathbb{Z}$. The matrix $A = (a_{ij})$ is invertible. Indeed, since the foliation $\mathcal{F}_\phi$ is minimal, the real numbers $v^{(1)}, \dots, v^{(n)}$ are linearly independent over $\mathbb{Q}$. So do the numbers $\lambda_\phi v^{(1)}, \dots, \lambda_\phi v^{(n)}$, which therefore can be taken for a basis of the module $\mathfrak{m}_\phi$. Thus, there exists an integer matrix $B = (b_{ij})$, such that $v^{(j)} = \sum_{i,j} w^{(i)}$, where $w^{(i)} = \lambda_\phi v^{(i)}$. Clearly, B is an inverse to the matrix A. Therefore, $A \in GL_n(\mathbb{Z})$.

Moreover, without loss of the generality one can assume that $a_{ij} \geq 0$. Indeed, if it is not yet the case, consider the conjugacy class $[A]$ of the matrix A. It is known that there exists a matrix $A^+ \in [A]$, whose entries are the non-negative integers. One has to replace the basis $v = (v^{(1)}, \dots, v^{(n)})$ in the module $\mathfrak{m}_\phi$ by the basis Tv, where $A^+ = TAT^{-1}$. It will be further assumed that $A = A^+$.

Lemma 5 *The vector $(v^{(1)}, \dots, v^{(n)})$ is the limit of a periodic Jacobi-Perron continued fraction.*

Proof. It follows from the discussion above, that there exists a non-negative integer matrix A, such that $Av = \lambda_\phi v$. In view of the Proposition 3 of [1], the matrix A admits a unique factorization:

$$A = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}, \tag{3}$$

where $b_i = (b_1^{(i)}, \dots, b_n^{(i)})^T$ are the vectors of the non-negative integers. Let us consider a periodic Jacobi-Perron continued fraction:

$$Per\ \overline{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}. \tag{4}$$

According to **Satz XII** of [12], the above periodic fraction converges to the vector $w = (w^{(1)}, \ldots, w^{(n)})$, such that w satisfies the equation $(B_1 B_2 \ldots B_k)w = Aw = \lambda_\phi w$. In view of the equation $Av = \lambda_\phi v$, we conclude that the vectors v and w are the collinear vectors. Therefore, the Jacobi-Perron continued fractions of v and w must coincide. $\square$

It is now straightforward to prove, that the AF-algebra attached to the foliation $\mathcal{F}_\phi$ is a stationary AF-algebra. Indeed, by the lemma 5, the vector of periods $v^{(i)} = \int_{\gamma_i} \omega$ unfolds into a periodic Jacobi-Perron continued fraction. By the definition, the Bratteli diagram of the AF-algebra $\mathbb{A}_\phi$ is periodic as well. In other words, the AF-algebra $\mathbb{A}_\phi$ is a stationary AF-algebra. $\square$

4.2 Proof of Theorem 2

(i) Let us prove the first statement. For the sake of completeness, let us give a proof of the following (well-known) lemma.

Lemma 6 *Let ϕ and ϕ' be the conjugate pseudo-Anosov automorphisms of a surface X. Then their invariant foliations $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi'}$ are the equivalent measured foliations.*

Proof. Let $\phi, \phi' \in Mod\ (X)$ be conjugate, i.e $\phi' = h \circ \phi \circ h^{-1}$ for an automorphism $h \in Mod\ (X)$. Since ϕ is the pseudo-Anosov automorphism, there exists a measured foliation $\mathcal{F}_\phi$, such that $\phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi$. Let us evaluate the automorphism ϕ' on the foliation $h(\mathcal{F}_\phi)$:

$$\begin{aligned} \phi'(h(\mathcal{F}_\phi)) &= h\phi h^{-1}(h(\mathcal{F}_\phi)) &&= h\phi(\mathcal{F}_\phi) = \\ &= h\lambda_\phi \mathcal{F}_\phi &&= \lambda_\phi(h(\mathcal{F}_\phi)). \end{aligned} \tag{5}$$

Thus, $\mathcal{F}_{\phi'} = h(\mathcal{F}_\phi)$ is the invariant foliation for the pseudo-Anosov automorphism ϕ' and $\mathcal{F}_\phi, \mathcal{F}_{\phi'}$ are equivalent foliations. Note also that the pseudo-Anosov automorphism ϕ' has the same dilatation as the automorphism ϕ. $\square$

Suppose that $\phi, \phi' \in Mod\ (X)$ are the conjugate pseudo-Anosov automorphisms. The functor F acts by the formulas $\phi \mapsto \mathbb{A}_\phi$ and $\phi' \mapsto \mathbb{A}_{\phi'}$, where $\mathbb{A}_\phi, \mathbb{A}_{\phi'}$ are the AF-algebras of the invariant foliations $\mathcal{F}_\phi, \mathcal{F}_{\phi'}$. In view of lemma 6, $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi'}$ are the equivalent measured foliations. Then, by lemma 4, the AF-algebras $\mathbb{A}_\phi$ and $\mathbb{A}_{\phi'}$ are the stably isomorphic AF-algebras. The item (i) follows.

(ii) Let us prove the second statement. We start with an elementary observation. Let $\phi \in Mod\ (X)$ be a pseudo-Anosov automorphism. Then there exists a unique measured foliation, $\mathcal{F}_\phi$, such that $\phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi$, where $\lambda_\phi > 1$

is an algebraic integer. Let us evaluate the automorphism $\phi^2 \in Mod\ (X)$ on the foliation $\mathcal{F}_\phi$:

$$\begin{aligned} \phi^2(\mathcal{F}_\phi) &= \phi(\phi(\mathcal{F}_\phi)) &= \phi(\lambda_\phi \mathcal{F}_\phi) = \\ = \lambda_\phi \phi(\mathcal{F}_\phi) &= \lambda_\phi^2 \mathcal{F}_\phi &= \lambda_{\phi^2} \mathcal{F}_\phi, \end{aligned} \tag{6}$$

where $\lambda_{\phi^2} := \lambda_\phi^2$. Thus, the foliation $\mathcal{F}_\phi$ is an invariant foliation for the automorphism ϕ^2 as well. By induction, one concludes that $\mathcal{F}_\phi$ is an invariant foliation for the automorphism ϕ^n for any $n \geq 1$.

Even more is true. Suppose that $\psi \in Mod\ (X)$ is a pseudo-Anosov automorphism, such that $\psi^m = \phi^n$ for some $m \geq 1$ and $\psi \neq \phi$. Then $\mathcal{F}_\phi$ is an invariant foliation for the automorphism ψ. Indeed, $\mathcal{F}_\phi$ is the invariant foliation for the automorphism ψ^m. If there exists $\mathcal{F}' \neq \mathcal{F}_\phi$ such that the foliation $\mathcal{F}'$ is an invariant foliation of ψ, then the foliation $\mathcal{F}'$ is also an invariant foliation of the pseudo-Anosov automorphism ψ^m. Thus, by the uniqueness, $\mathcal{F}' = \mathcal{F}_\phi$. We have just proved the following lemma.

Lemma 7 *Let ϕ be the pseudo-Anosov automorphism of a surface X. Denote by $[\phi]$ a set of the pseudo-Anosov automorphisms ψ of X, such that $\psi^m = \phi^n$ for some positive integers m and n. Then the pseudo-Anosov foliation $\mathcal{F}_\phi$ is an invariant foliation for every pseudo-Anosov automorphism $\psi \in [\phi]$.*

In view of lemma 7, one arrives at the following identities among the AF-algebras:

$$\mathbb{A}_\phi = \mathbb{A}_{\phi^2} = \ldots = \mathbb{A}_{\phi^n} = \mathbb{A}_{\psi^m} = \ldots = \mathbb{A}_{\psi^2} = \mathbb{A}_\psi. \tag{7}$$

Thus, the functor F is not an injective functor: the preimage, $Ker\ F$, of the AF-algbera $\mathbb{A}_\phi$ consists of a countable set of the pseudo-Anosov automorphisms $\psi \in [\phi]$, commensurable with the automorphism ϕ.

Theorem 2 is proved. $\square$

4.3 Proof of Corollary 1

(i) It follows from theorem 1, that $\mathbb{A}_\phi$ is a stationary AF-algebra. An arithmetic invariant of the stable isomorphism classes of the stationary AF-algebras has been found by D. Handelman in [7]. Summing up his results, the invariant is as follows.

Let $A \in GL_n(\mathbb{Z})$ be a matrix with the strictly positive entries, such that A is equal to the minimal period of the Bratteli diagram of the stationary AF-algebra. (In case the matrix A has the zero entries, it is necessary to take a proper minimal power of the matrix A.) By the Perron-Frobenius theory, the matrix A has a real eigenvalue $\lambda_A > 1$, which exceeds the absolute values of other roots of the characteristic polynomial of A. Note that λ_A is an algebraic integer (unit). Consider a real algebraic number field $K = \mathbb{Q}(\lambda_A)$ obtained as

an extension of the field of the rational numbers by the algebraic number λ_A. Let $(v_A^{(1)}, \dots, v_A^{(n)})$ be the eigenvector corresponding to the eigenvalue λ_A. One can normalize the eigenvector so that $v_A^{(i)} \in K$.

The departure point of Handelman's invariant is the $\mathbb{Z}$-module $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \dots + \mathbb{Z}v_A^{(n)}$. The module $\mathfrak{m}$ brings in two new arithmetic objects: (i) the ring Λ of the endomorphisms of $\mathfrak{m}$ and (ii) an ideal I in the ring Λ, such that $I = \mathfrak{m}$ after a scaling ([4], Lemma 1, p.88). The ring Λ is an order in the algebraic number field K and therefore one can talk about the ideal classes in Λ. The ideal class of I is denoted by $[I]$. Omitting the embedding question for the field K, the triple $(\Lambda, [I], K)$ is an invariant of the stable isomorphism class of the stationary AF-algebra $\mathbb{A}_\phi$ (§5 of [7]). The item (i) follows.

(ii) The numerical invariants of the stable isomorphism classes of the stationary AF-algebras can be derived from the triple $(\Lambda, [I], K)$. These invariants are the rational integers – called the determinant and signature – can be obtained as follows.

Let $\mathfrak{m}, \mathfrak{m}'$ be the full $\mathbb{Z}$-modules in an algebraic number field K. It follows from (i), that if $\mathfrak{m} \neq \mathfrak{m}'$ are distinct as the $\mathbb{Z}$-modules, then the corresponding AF-algebras cannot be stably isomorphic. We wish to find the numerical invariants, which discern the case $\mathfrak{m} \neq \mathfrak{m}'$. It is assumed that a $\mathbb{Z}$-module is given by the set of generators $\{\lambda_1, \dots, \lambda_n\}$. Therefore, the problem can be formulated as follows: find a number attached to the set of generators $\{\lambda_1, \dots, \lambda_n\}$, which does not change on the set of generators $\{\lambda_1', \dots, \lambda_n'\}$ of the same Z-module.

One such invariant is associated with the trace function on the algebraic number field K. Recall that $Tr : K \to \mathbb{Q}$ is a linear function on K such that $Tr\ (\alpha + \beta) = Tr\ (\alpha) + Tr\ (\beta)$ and $Tr\ (a\alpha) = a\ Tr\ (\alpha)$ for $\forall \alpha, \beta \in K$ and $\forall a \in \mathbb{Q}$.

Let $\mathfrak{m}$ be a full $\mathbb{Z}$-module in the field K. The trace function defines a symmetric bilinear form $q(x,y) : \mathfrak{m} \times \mathfrak{m} \to \mathbb{Q}$ by the formula:

$$(x, y) \longmapsto Tr\ (xy), \qquad \forall x, y \in \mathfrak{m}. \tag{8}$$

The form $q(x,y)$ depends on the basis $\{\lambda_1, \dots, \lambda_n\}$ in the module $\mathfrak{m}$:

$$q(x,y) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i y_j, \qquad \text{where} \quad a_{ij} = Tr\ (\lambda_i \lambda_j). \tag{9}$$

However, the general theory of the bilinear forms (over the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ or the ring of rational integers $\mathbb{Z}$) tells us that certain numerical quantities will not depend on the choice of such a basis.

Namely, one such invariant is as follows. Consider a symmetric matrix A

corresponding to the bilinear form $q(x,y)$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}. \tag{10}$$

It is known that the matrix A, written in a new basis, will take the form $A' = U^T A U$, where $U \in GL_n(\mathbb{Z})$. Then $det\ (A') = det\ (U^T A U) = = det\ (U^T) det\ (A) det\ (U) = det\ (A)$. Therefore, the rational integer number:

$$\Delta = det\ (Tr\ (\lambda_i \lambda_j)), \tag{11}$$

called a *determinant* of the bilinear form $q(x,y)$, does not depend on the choice of the basis $\{\lambda_1, \dots, \lambda_n\}$ in the module $\mathfrak{m}$. We conclude that the determinant Δ discerns [1] the modules $\mathfrak{m} \neq \mathfrak{m}'$.

Finally, recall that the form $q(x,y)$ can be brought by the integer linear substitutions to the diagonal form:

$$a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2, \tag{12}$$

where $a_i \in \mathbb{Z} - \{0\}$. We let a_i^+ be the positive and a_i^- the negative entries in the diagonal form. In view of the law of inertia for the bilinear forms, the integer number $\Sigma = (\# a_i^+) - (\# a_i^-)$, called a *signature*, does not depend on a particular choice of the basis in the module $\mathfrak{m}$. Thus, Σ discerns the modules $\mathfrak{m} \neq \mathfrak{m}'$. The corollary 1 follows. $\square$

5 Examples, open problems and conjectures

In the present section we shall calculate the invariants Δ and Σ for the Anosov automorphisms of the two-dimensional torus. An example of two non-conjugate Anosov automorphisms with the same Alexander polynomial, but different determinants Δ is constructed. Recall that the isotopy classes of the orientation-preserving diffeomorphisms of the torus T^2 are bijective with the 2×2 matrices with integer entries and determinant $+1$, i.e. $Mod\ (T^2) \cong SL(2, \mathbb{Z})$. Under the identification, the non-periodic automorphisms correspond to the matrices $A \in SL(2, \mathbb{Z})$ with $|Tr\ A| > 2$.

[1]Note that if $\Delta = \Delta'$ for the modules $\mathfrak{m}, \mathfrak{m}'$, one cannot conclude that $\mathfrak{m} = \mathfrak{m}'$. The problem of equivalence of the symmetric bilinear forms over $\mathbb{Q}$ (i.e. the existence of a linear substitution over $\mathbb{Q}$, which transforms one form to the other), is a fundamental question of number theory. The Minkowski-Hasse theorem says that two such forms are equivalent if and only if they are equivalent over the field $\mathbb{Q}_p$ for every prime number p and over the field $\mathbb{R}$. Clearly, the resulting p-adic quantities will give new invariants of the stable isomorphism classes of the AF-algebras. The question is much similar to the Minkowski units attached to knots, see e.g. Reidemeister [13]. We will not pursue this topic here and refer the reader to the problem part of present article.

5.1 Full modules and orders in the quadratic fields

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension of the field of rational numbers $\mathbb{Q}$. Further we suppose that d is a positive square free integer. Let

$$\omega = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \ mod\ 4, \\ \sqrt{d} & \text{if } d \equiv 2, 3 \ mod\ 4. \end{cases} \tag{13}$$

Proposition 1 *Let f be a positive integer. Every order in K has form $\Lambda_f = \mathbb{Z} + (f\omega)\mathbb{Z}$, where f is the conductor of Λ_f.*

Proof. See [4] pp. 130-132. □

The proposition 1 allows to classify the similarity classes of the full modules in the field K. Indeed, there exists a finite number of $\mathfrak{m}_f^{(1)}, \dots, \mathfrak{m}_f^{(s)}$ of the non-similar full modules in the field K, whose coefficient ring is the order Λ_f, cf Theorem 3, Ch 2.7 of [4]. Thus, proposition 1 gives a finite-to-one classification of the similarity classes of full modules in the field K.

5.2 The numerical invariants of the Anosov automorphisms

Let Λ_f be an order in K with the conductor f. Under the addition operation, the order Λ_f is a full module, which we denote by $\mathfrak{m}_f$. Let us evaluate the invariants $q(x, y)$, Δ and Σ on the module $\mathfrak{m}_f$. To calculate $(a_{ij}) = Tr(\lambda_i \lambda_j)$, we let $\lambda_1 = 1, \lambda_2 = f\omega$. Then:

$$\begin{aligned} a_{11} &= 2, \quad a_{12} = a_{21} = f, \quad a_{22} = \frac{1}{2} f^2 (d+1) \quad \text{if} \quad d \equiv 1 \ mod\ 4 \\ a_{11} &= 2, \quad a_{12} = a_{21} = 0, \quad a_{22} = 2f^2 d \quad \text{if} \quad d \equiv 2, 3 \ mod\ 4, \end{aligned} \tag{14}$$

and

$$\begin{aligned} q(x, y) &= 2x^2 + 2fxy + \frac{1}{2} f^2 (d+1) y^2 \quad \text{if} \quad d \equiv 1 \ mod\ 4 \\ q(x, y) &= 2x^2 + 2f^2 d y^2 \quad \text{if} \quad d \equiv 2, 3 \ mod\ 4. \end{aligned} \tag{15}$$

Therefore

$$\Delta = \begin{cases} f^2 d & \text{if } d \equiv 1 \ mod\ 4, \\ 4f^2 d & \text{if } d \equiv 2, 3 \ mod\ 4, \end{cases} \tag{16}$$

and $\Sigma = +2$ in the both cases, where $\Sigma = \#(positive) - \#(negative)$ entries in the diagonal normal form of $q(x, y)$.

5.3 An example

Let us consider a numerical example, which illustrates the advantages of our invariants in comparison to the classical Alexander polynomials. Denote by M_A

and M_B the hyperbolic 3-dimensional manifolds obtained as a torus bundle over the circle with the monodromies

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}, \tag{17}$$

respectively. The Alexander polynomials of M_A and M_B are identical $\Delta_A(t) = \Delta_B(t) = t^2 - 6t + 1$. However, the manifolds M_A and M_B are *not* homotopy equivalent.

Indeed, the Perron-Frobenius eigenvector of matrix A is $v_A = (1, \sqrt{2} - 1)$ while of the matrix B is $v_B = (1, 2\sqrt{2} - 2)$. The bilinear forms for the modules $\mathfrak{m}_A = \mathbb{Z} + (\sqrt{2} - 1)\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (2\sqrt{2} - 2)\mathbb{Z}$ can be written as

$$q_A(x, y) = 2x^2 - 4xy + 6y^2, \qquad q_B(x, y) = 2x^2 - 8xy + 24y^2, \tag{18}$$

respectively. The modules $\mathfrak{m}_A, \mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{2})$, since their determinants $\Delta(\mathfrak{m}_A) = 8$ and $\Delta(\mathfrak{m}_B) = 32$ are not equal. Therefore the matrices A and B are not conjugate [2] in the group $SL(2, \mathbb{Z})$. Note that the class number $h_K = 1$ for the field K.

5.4 The open problems and conjectures

The section is reserved for some questions and conjectures, which arise in the connection with the invariants $(\Lambda, [I], K), q(x, y), \Delta$ and Σ.

1. The p-adic invariants of pseudo-Anosov automorphisms

A. Let $\phi \in Mod\ (X)$ be pseudo-Anosov automorphism of a surface X. If λ_ϕ is the dilatation of ϕ, then one can consider a $\mathbb{Z}$-module $\mathfrak{m} = \mathbb{Z}v^{(1)} + \ldots + \mathbb{Z}v^{(n)}$ in the number field $K = \mathbb{Q}(\lambda_\phi)$ generated by the normalized eigenvector $(v^{(1)}, \ldots, v^{(n)})$ corresponding to the eigenvalue λ_ϕ. The trace function on the number field K gives rise to a symmetric bilinear form $q(x, y)$ on the module $\mathfrak{m}$. The form is defined over the field $\mathbb{Q}$. It has been shown that a pseudo-Anosov automorphism ϕ', conjugate to ϕ, yields a form $q'(x, y)$, equivalent to $q(x, y)$, i.e. $q(x, y)$ can be transformed to $q'(x, y)$ by an invertible linear substitution with the coefficients in $\mathbb{Z}$.

B. Recall that in order the two rational bilinear forms $q(x, y), q'(x, y)$ to be equivalent, it is necessary and sufficient, that the following conditions were met:

(i) $\Delta = \Delta'$, where Δ is the determinant of the form;

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[2] The reader may verify this fact using the method of periods, which dates back to Gauss. First we have to find the fixed points $Ax = x$ and $Bx = x$, which gives us $x_A = 1 + \sqrt{2}$ and $x_B = \frac{1+\sqrt{2}}{2}$, respectively. Then one unfolds the fixed points into a periodic continued fraction, which gives us $x_A = [2, 2, 2, \ldots]$ and $x_B = [1, 4, 1, 4, \ldots]$. Since the period (2) of x_A differs from the period $(1, 4)$ of B, the matrices A and B belong to different conjugacy classes in $SL(2, \mathbb{Z})$.

(ii) for each prime number p (including $p = \infty$) certain p-adic equation between the coefficients of forms q, q' must be satisfied, see e.g. [4], Ch.1, §7.5. (In fact, only a *finite* number of such equations have to be verified.)

The condition (i) has been already used to discern between the conjugacy classes of the pseudo-Anosov automorphisms. One can use condition (ii) to discern between the pseudo-Anosov automorphisms with $\Delta = \Delta'$. The following question can be posed: *Find the p-adic invariants of the pseudo-Anosov automorphisms.*

2. The signature of a pseudo-Anosov automorphism

The signature is an important and well-known invariant connected to the chirality and knotting number of knots and links [13]. It will be interesting to find a geometric interpretation to the signature Σ for the pseudo-Anosov automorphisms. One can ask the following question: *Find a geometric meaning of the invariant Σ.*

3. The number of the conjugacy classes of pseudo-Anosov automorphisms with the same dilatation

The dilatation λ_ϕ is an invariant of the conjugacy class of the pseudo-Anosov automorphism $\phi \in Mod\ (X)$. On the other hand, it is known that there exist non-conjugate pseudo-Anosov's with the same dilatation and the number of such classes is finite [15]. It is natural to expect that the invariants of operator algebras can be used to evaluate the number. We conclude with the following conjecture.

Conjecture 1 *Let $(\Lambda, [I], K)$ be the triple corresponding to a pseudo-Anosov automorphism $\phi \in Mod\ (X)$. Then the number of the conjugacy classes of the pseudo-Anosov automorphisms with the dilatation λ_ϕ is equal to the class number $h_\Lambda = |\Lambda/[I]|$ of the integral order Λ.*

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